Recap

- crit. pts, reg. pts
  crit. value, reg. value of $f: M \to N$ smooth

- corank ~ defect of df at x

  $\min(m, n) - \operatorname{rank} df_x$

  $m \geq n$ corank $= \dim \operatorname{coker} df_x$

  $m \leq n$ " $= \dim \ker df_x$

- equivalence $\leftrightarrow$ l-r action

  $f \sim g \iff g = l \circ f \circ r^{-1}$

- stability

  "every small perturbation is equiv."

- germ of a map at a point (subset)

  $\to$ equivalence & stability of germs

  !

  Q: can formulate via a group action of diffeom. germs ?

Smooth maps & crit. pts x

$\downarrow$

germs of maps at x

$\downarrow$

Jets: k-jet of f at x $\sim$ Taylorpolynomial of f at x

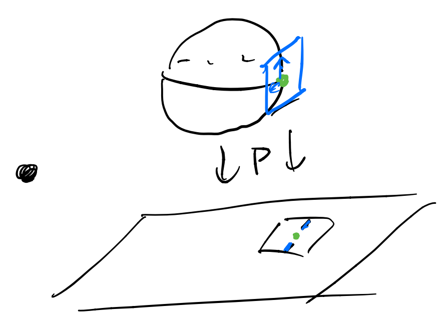
More later, first we restrict attention to the first derivative.

II. The classes Σ^I

7. Def.: $\Sigma^i(f) = \{ x \in M \mid \operatorname{corank}(df_x) = i \}$

Note: $\Sigma^0(f) = $ reg. pts.

e.g.



$\{\text{equator}\} = \Sigma^1(p)$

$S^2 \setminus \{\text{eq.}\} = \Sigma^0(p)$

- $f(z) = z^2$ as <u>smooth</u> map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \qquad df_{(x,y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\Rightarrow \quad \Sigma^2(f) = \{0\}$$
$$\Sigma^1(f) = \emptyset \qquad \Sigma^0(f) = \mathbb{R}^2 \setminus \{0\}$$

Q: <u>Stable?</u>

$\rightsquigarrow$ Do the $\Sigma^i(f)$ give us a decomposition of M?

<u>8. Thm.:</u> For a "generic" map $f: M \to N$ the sets $\Sigma^i(f)$ are submanifolds of M with

$$\operatorname{codim} \Sigma^i(f) := \dim M - \dim \Sigma^i(f)$$
$$= \underset{\substack{\uparrow \\ \text{corank of } f}}{i} \cdot \big(\underbrace{|\dim N - \dim M| + i}_{\text{"corank at the target"}} \big)$$
$$= (\dim M - r) \cdot (\dim N - r) \quad \text{w. } r = \operatorname{rank} df$$

(if this nr. is neg., then $\Sigma^i(f) = \emptyset$).

We put this theorem to use by simple dimension-counts:

$\longrightarrow$ $z \mapsto z^2$ is generic as holomorphic map, but not as a smooth map.

$\longrightarrow$ generic smooth maps from $\mathbb{R}^m \to \mathbb{R}^n$ have $\Sigma^m(f) = \emptyset$, more generally

$$\Sigma^i(f) = m - i^2$$

To prove the theorem we look first at the linear case:

Consider $\mathcal{L}_{mn} := \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n) \cong \{A \in M_{n,m}(\mathbb{R})\} \cong \mathbb{R}^{m \cdot n}$

left-right action of $Gl_m \times Gl_n$ on $\mathcal{L}_{mn}$ (drop the "$\mathbb{R}$")

$\forall A \in M_{nm}$ $\exists R \in Gl_m, L \in Gl_n$

s.t. $LAR^{-1} = A_0 := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

where $r = \operatorname{rank}(A)$

9. Lemma The set of all matrices of rank r

$M^r_{n\cdot m} = \{ A \in M_{nm} \mid \operatorname{rank}(A) = r \} \subset M_{nm}$ is

a submanifold with $\operatorname{codim} M^r_{nm} = (m-r)\cdot(n-r)$
$= i\,(|n-m| + i)$

Proof

Fix r and let $A \in M^r_{nm}$. After changing coordinates we can assume

$$A = \begin{array}{c c} & \begin{array}{cc} r & m-r \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \left(\begin{array}{c:c} B & C \\ \hdashline D & E \end{array}\right) \begin{array}{c} n \\ n-r \end{array} \\ & \begin{array}{cc} & m-r \end{array} \end{array} \quad \text{with } B \in GL_r$$

right-multiply by $\begin{pmatrix} I_r & -B^{-1}C \\ 0 & I_{m-r} \end{pmatrix}$ gives

$$\begin{pmatrix} B & 0 \\ 0 & E - DB^{-1}C \end{pmatrix} \quad \text{which has rank } r$$

$$\Longleftrightarrow \quad E = DB^{-1}C \in M_{n-r,m-r}$$

Now define $F: M_{n,m} \to M_{n-r,m-r}$

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \mapsto E - DB^{-1}C$$

and observe

- F smooth
- $0 \in M_{n-r,m-r}$ is a reg. value

[for $X \in M_{n-r,m-r}$ consider $\gamma(t) = \begin{pmatrix} B & C \\ D & E + t \cdot X \end{pmatrix}$

$\dot{\gamma}(0) \overset{dF}{\mapsto} X$]

preimage thm $\Longrightarrow$ $F^{-1}(0) = M^r_{nm}$ is a submf of M_{nm} of codim $M^r_{nm} = \dim M_{n-r,m-r}$

$= (n-r)(m-r)$

$\blacksquare$

Our goal: Lift Lemma 9 to Theorem 8.

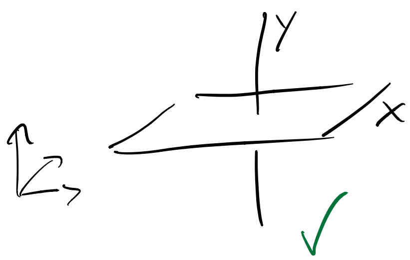
For this we need the notion of <u>transversality</u> ("opposite of tangency")

10. Def.:

1. V vect. space, X, Y lin. subspaces are called transversal (in V), $X \pitchfork Y$, if

$$V = X + Y \quad (\text{or } X \cap Y = \emptyset)$$

e.g.:



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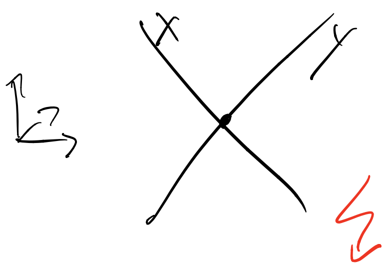
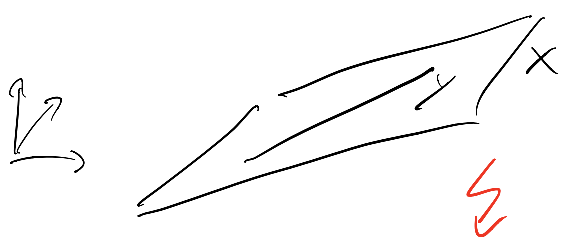




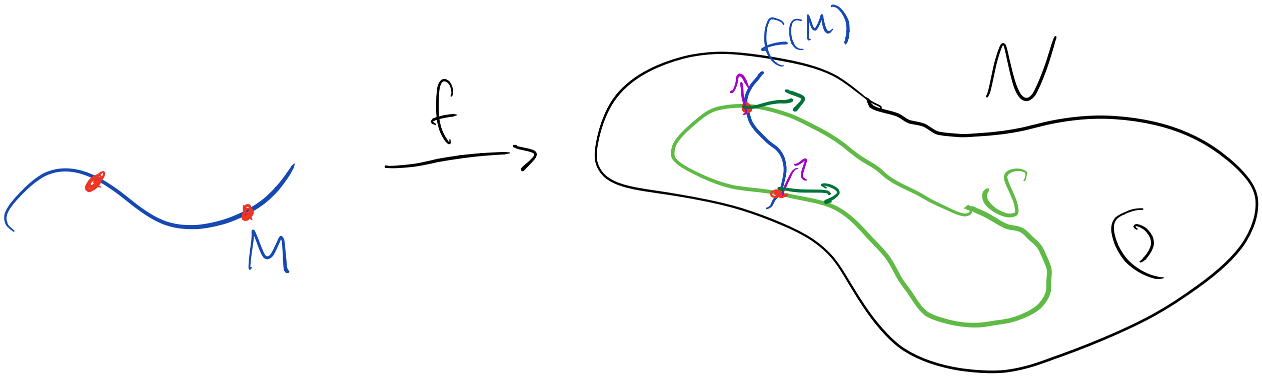
2. $f: M \to N$ smooth, $S \subset N$ submf.

f is transversal to S at $x \in M$

if either $f(x) \notin S$

or $df_x(T_xM) + T_{f(x)}S = T_{f(x)}N$

f is transversal to S, $f \pitchfork S$, if this holds for all $x \in M$.

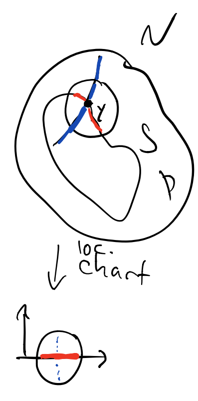


11. Prop.:

If $f \pitchfork S$, then $f^{-1}(S)$ is a subm.f of M with $\operatorname{codim} f^{-1}(S) = \operatorname{codim} S$.

Proof: Find $F: M \longrightarrow \mathbb{R}^{n-k}$ ($k = \dim S$) with $F^{-1}(0) = f^{-1}(S)$ and use preimage theorem.

Let $y \in S$. S submf $\Rightarrow$ locally there are coordinates $(y_1, \dots, y_k, y_{k+1}, \dots, y_n)$ s.t. $S = (y_1, \dots, y_k, \overset{\in \mathbb{R}^{n-k}}{0})$



Define $g: N \to \mathbb{R}^{n-k}$, $(y_1, \dots, y_n) \mapsto (y_{k+1}, \dots, y_n)$

Then g is smooth with $S = g^{-1}(0)$

and $dg_{(y_1 \dots y_n)}$ surjective.

preimage thm
$\Rightarrow$

$$\ker dg_y = T_y S \text{ ,}$$

Define $F := g \circ f : M \to \mathbb{R}^{n-k}$. Then F is smooth, $F^{-1}(0) = f^{-1}(S)$ and DF_x is surjective for all $x \in f^{-1}(S)$ by transversality:

$$f \pitchfork S : \quad df_x(T_x M) + \underset{\ker dg_{f(x)} \, !}{\underset{\parallel}{T_{f(x)} S}} = T_{f(x)} N$$

"apply" $dg_{f(x)}$ to get

$$\underset{dF_x(T_x M)}{\underset{\parallel}{dg_{f(x)} df_x (T_x M)}} + dg_{f(x)}(\ker dg_{f(x)})$$

$$= dg_{f(x)}(T_{f(x)} N)$$

$$= \mathbb{R}^{n-k}$$

We're getting closer to an idea of the term "generic" ...

12 Thm: (weak transversality theorem)

For M closed, $S \subset N$ closed, the set of maps transversal to S,

$$C^\infty_{\pitchfork S}(M,N) = \{ f \in \underline{C^\infty(M,N)} \mid f \pitchfork S \}$$

forms an open dense set in $C^\infty(M,N)$, both in the weak and strong topology.

Roughly:

strong $\sim$ $U_\varepsilon(f) = \{ g \in C^\infty(M,N) \mid \|D^k f - D^k g\|_\infty < \varepsilon$ for all $k \in \mathbb{N} \}$

weak $\sim$ same but condition reduced to hold for compact sets only.

More on these topologies later (or see the book "Differential Topology" by Hirsch)